Exam — Introduction to Optimization
Friday, November 8, 2024
University of Groningen

Let $f : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ be a closed convex function with minimizers. Consider the problem:

(1) Find *the* minimizer of f that has the smallest norm.

The problem is well posed, in the sense that f does have a unique least-norm minimizer. We denote it by $\hat{x}$. You do not need to prove this. All the questions that follow have the same weight.

Part I. Given $\varepsilon > 0$, define $f_\varepsilon : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ by

$$f_\varepsilon(x) = f(x) + \frac{\varepsilon}{2}\|x\|^2.$$

1. Show that f_ε is closed and ε-strongly convex, whence it has a unique minimizer, which we denote by x_ε.
2. Verify that $\min(f) + \frac{\varepsilon}{2}\|x_\varepsilon\|^2 \leq f_\varepsilon(x_\varepsilon) \leq f_\varepsilon(x)$ for every $x \in \mathbb{R}^N$.
3. Prove that $\|x_\varepsilon\| \leq \|\hat{x}\|$ for every $\varepsilon > 0$ (recall that $\hat{x}$ is the minimizer of f that has the smallest norm).
4. Verify that $\lim_{\varepsilon \to 0} f(x_\varepsilon) = \min(f)$.
5. Use 3 and 4 to conclude that $x_\varepsilon \to \hat{x}$ as $\varepsilon \to 0$.

Part II. Let (ε_k) be a positive real sequence such that $\varepsilon_k \to 0$ as $k \to \infty$. Pick $\gamma > 0$ and $x_0 \in \mathbb{R}^N$, and define a sequence (x_k) by iterating

$$x_{k+1} = \text{prox}_{\gamma f_{\varepsilon_k}}(x_k) = \text{argmin}\left\{ f_{\varepsilon_k}(x) + \frac{1}{2\gamma}\|x - x_k\|^2 \right\}, \tag{2}$$

for $k \geq 0$. The purpose of this part is to show that this procedure converges to the solution of (1).

6. Write the optimality condition for (2).
7. Use 1 and 6 to show that

$$f_{\varepsilon_k}(\hat{x}) \geq f_{\varepsilon_k}(x_{k+1}) - \frac{1}{\gamma}(x_{k+1} - x_k) \cdot (\hat{x} - x_{k+1}) + \frac{\varepsilon_k}{2}\|x_{k+1} - \hat{x}\|^2.$$

8. Now, use 2 and 7 to prove that

$$(1 + \gamma\varepsilon_k)\|x_{k+1} - \hat{x}\|^2 \leq \|x_k - \hat{x}\|^2 + \gamma\varepsilon_k\left[\|\hat{x}\|^2 - \|x_{\varepsilon_k}\|^2\right].$$

9. Use 3, 5 and the Lemma below (which you do not need to prove) to conclude that, if $\sum_{k\geq 0} \varepsilon_k = \infty$, then $x_k \to \hat{x}$ as $k \to \infty$.

Lemma. Let (A_k), (h_k) and (δ_k) be nonnegative real sequences such that $(1 + \delta_k)A_{k+1} \leq A_k + \delta_k h_k$ for every $k \geq 0$. If $h_k \to 0$ as $k \to \infty$ and $\sum_{k\geq 0} \delta_k = \infty$, then $A_k \to 0$ as $k \to \infty$.